used as data, of the same form as those in (17) and (28), are all multiplied by a constant number.

The positions of the free axes in the final product of any number of different polynomial factors will be given by the same formulas as above, when for the moments used we substitute the sums obtained by adding together the corresponding moments for all the factors.

When the free axes are taken as coordinate axes, simple properties like those we have already found can be demonstrated for moments of still higher orders. By the same process of differentiation and multiplication, and then applying (6), (13) and (35), it is found that

$$\Sigma_3(lx^3y) = \Sigma_3(Lx^3y)+\Sigma_3(L'x^3y), \tag{37}$$

the like being true of course for the x^3z, xy^3, y^3z, xz^3 and yz^3 moments. It is also found that

$$\Sigma_3(lx^2yz) = \Sigma_3(Lx^2yz)+\Sigma_3(L'x^2yz), \tag{38}$$

the same being true of the xy^2z and xyz^2 moments.

For the form x^2y^2 the relation is more complex, namely

$$\Sigma_3(lx^2y^2) = \Sigma_3(Lx^2y^2)+\Sigma_3(L'x^2y^2)+\Sigma_3(Lx^2)\Sigma_3(L'y^2) +\Sigma_3(Ly^2)\Sigma_3(L'x^2). \tag{39}$$

The like holds true for the x^2z^2 and y^2z^2 moments.

CORRESPONDENCE.

Editor Analyst:

If not trespassing too much upon your space permit me to reply in a few words to the letters of Mr. Adcock and Prof. Judson, in the last number of the ANALYST.

Mr. Adcock, after quoting from my original statement of the paradox, "now if $a=\infty$, $u=0$ independently of x", adds "This I deny." His first answer begins, "When $u=0$, independently of x it is not a function of x", from which it seems that he was at that time content to accept the equation $u=0$. It is difficult to see what he wishes now to substitute for this eq'n, for he goes on to say, "In this case, $u=$ actual zero, or an infinitesimal." Perhaps he means only to deny t e clause "independently of x"; for he remarks "the rate at which these infinitesimals change their value is $du \div dx = \cos ax$." But this is a mere restatement of the paradox; viz., that u has a finite rate of change, and yet no finite change in value.

Mr. Adcock's "private interpretation" of the form $\cos\infty$, as "indetermi-

nate both in form and value," agrees with my own, as expressed in my last letter as well as in the original statement of the paradox; viz., that cos ∞ is an "*essentially* indeterminate form." Neverthelss he will not deny that there is some interest attaching to De Morgan's "private interpretation" of the same form.

In this connection, it should be stated, in reply to Prof. Judson's remark that we cannot admit zero as the value of sin ∞ and cos ∞ "in violation of the principle $\sin^2\infty+\cos^2\infty = 1$", that the writers who take this view expressly state that, by their principles of interpretation, $\sin^2\infty$ is not to be regarded as the square of sin ∞, but as the mean value of $\sin^2 x$, which is $\frac{1}{2}$.

Again, De Morgan finds $\sec\infty = 0$ (not equal to the reciprocal of $\cos\infty$) and $\tan\infty = \sqrt{-1}$; but he appears to violate these principles by citing as a verification of these last values the fact that they satisfy the equation $\sec^2\infty = \tan^2\infty + 1$.

Prof. Judson states very clearly, in his last letter, the "principle which seems to be violated in the example given," i. e. that, when we assign to a such a value as will reduce it to a constant or to zero then the same value ought to reduce the derivative with respect to x to zero. This is certainly what we naturally expect, and what we usually find to be the case. Nevertheless I maintain that the principle as stated not only "seems to be", but *is* violated, in opposition to Prof. Judson's opinion that the paradox must arise from a misinterpretation of the form $a\div\infty$. In support of this Prof. Judson says that when $a = \infty u$ "becomes indefinite." This certainly is not true in the sense in which $du \div dx$ becomes indefinite, and therefore I suppose means infinitesimal, in which case we are told "it still remains a function of x and there seems to be no good reason why we should then expect $du\div dx$ to equal zero, *independently* of x." The phrase "it still remains a function of x" is here misleading; for u is no longer a function of x in the sense that its value changes with x by any finite amount, and it is simply the fact that it does not change its value by any finite amount that leads us to expect that its derivative should reduce to zero.

If we examine the principle as stated above, we shall I think find that though generally, it is not universally true, and shall see why it fails in the case in question. When a function has an actual (not zero) rate of change, the ratio of its increment to that of x has an actual limiting value which measures this rate, and is the derivative found by the ordinary rules of differentiation; whence we naturally assume (what is in fact generally true) the converse theorem; that when the rate is zero the derivative is also zero. But to prove this in any case, it is necessary to show that the difference between the actual increment of the function and the product of the derivative

by the increment of x (the second term in the expansion by Taylor's Theorem) vanishes in comparison with this product. This is not necessarily true when the second derivative is infinite, which is the case with the function in question. This is made clear by writing out the development, thus

$$\frac{\sin a(x+h)}{a} = \frac{\sin ax}{a} + \cos ax.h - a\sin ax.\frac{h^2}{2} - \ldots$$

The actual increment of the function, consisting of all the terms of the series after the first, is indeed zero, because the function does not change its value when x is replaced by $x+h$; but it is not true that the coefficent of h is the limit of

$$\left[\frac{\sin a(x+h)}{a} - \frac{\sin ax}{a}\right] \div h$$

because the result of dividing the terms which follow the second by h does not vanish with h.

Of course if with Mr. Todhunter and other writers we define the derivative or "first differential coefficient" as the limit of the ratio written above, we are compelled to reject the equation $du \div dx = \cos ax$ when $a = \infty$. If on the other hand we admit this equation, because it results from the rules of differentiation, we cannot deny that the first derivative is a measure of the rate, unless the higher derivatives are finite.

WM. WOOLSEY JOHNSON.

Editor Analyst:

I submit the following correction of an error in Bartlett's Mechanics, ninth edition, p. 398, equations (580).

From the equation

$$\frac{d^2\xi}{dt^2} = -n_x^2\xi,$$

the author deduces by integration

$$\frac{d\xi^2}{dt^2} = -n_x^2\xi^2,$$

whereas it should be

$$\frac{d\xi^2}{dt^2} = n_x^2(a_x^2 - \xi^2),$$

where a_x is the constant of integration. The corrected equation may be deduced directly from equations (530) of that work. The other two equat'ns of (580) should be corrected in the same manner.

DE VOLSON WOOD.